2015

(October)

MATHEMATICS

(Elective/Honours)

FIRST PAPER

(Algebra—I and Calculus—I)

Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **five** question, taking **one** from each Unit

UNIT—I

1. *(a)* If a finite set S has n elements, then prove that the power set of S has 2^n elements. 4

   (b) Show that the domain of definition of the function $f(x) = \log \frac{1-x}{1+x}$ is the interval $(-1, 1)$. Also, show that for $x_1, x_2 \in (-1, 1)$

$$f(x_1) + f(x_2) = f\left(\frac{x_1 + x_2}{1 + x_1 x_2}\right)$$ 2+3

(c) Draw the graph of the function

$$f(x) = \begin{cases} 0, & x = 0 \\ 1 - x, & 0 < x < 1 \\ 1, & x = 1 \end{cases}$$

The graph suggests that the function $f(x)$ assumes at least once every value between $f(0)$ and $f(1)$ and yet it has a discontinuity in $[0, 1]$. Justify this. 3+3

2. (a) Let $S = \{-2, -1, 0, 1, 2\}$. Give examples of the following : 2+2+2

(i) A relation on S that is reflexive but not transitive

(ii) A relation on S that is reflexive but not symmetric

(iii) A relation on S that is symmetric but not reflexive

(b) Using definition of limit, prove that

$$\lim_{x \to 2} 5x = 10$$ 3

(c) Discuss the continuity of the function $f(x) = \frac{|x|}{x}$ at $x = 0$ and $x = 1$. 3+3

UNIT—II

3. (a) Let $f : Q \to Q$ be defined by $f(x) = 2x + 3$, where Q = set of rational numbers. Show that f is one-to-one and onto. Also, find a formula that defines the inverse function f^{-1}. 3+2

(b) Give examples of—

(i) matrices A, B such that $AB \neq BA$;

(ii) matrices A, B such that $AB = 0$ but $A \neq 0$, $B \neq 0$. 2+2

(c) Determine if the following system of equations is consistent and if so, find the solution : 2+4

$$x - y + 2z = 4$$
$$3x + y + 4z = 6$$
$$x + y + z = 1$$

4. (a) If B is an idempotent matrix, show that $A = I - B$ is also idempotent and $AB = BA = 0$. 3+2

(b) Consider the equation

$$f(x) \equiv x^3 - 3x^2 + 4x - 3 = 0$$

Find $f(A)$ and A^{-1} if

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$$ 4

(c) Applying elementary row operation, find the rank of the matrix

$$\begin{pmatrix} -4 & 1 & -1 & 2 \\ 1 & 0 & -1 & 0 \\ -5 & 2 & -5 & 4 \end{pmatrix}$$ 6

UNIT—III

5. (a) Show that $f(x) = x^2$ for $-2 \le x \le 2$ is uniformly continuous. 4

(b) Find the slopes of the parabola $y = x^2$ at the vertex and at the point $\left(\frac{1}{2}, \frac{1}{4}\right)$. Determine whether the tangent line to the parabola at the point $\left(\frac{1}{2}, \frac{1}{4}\right)$ makes an angle 45° with x-axis with justification. 3+2

(c) If $y = \cos(m \sin^{-1} x)$, show that—

(i) $(1-x^2)y_2 - xy_1 + m^2 y = 0$;

(ii) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2 - n^2)y_n = 0$. 2+4

6. (a) If the rate of change of y with respect to x is 5 and x is changing at 3 units per second, how fast is y changing? 3

(b) Find $\frac{d^2y}{dx^2}$, where $y = x\sqrt{x^2+9}$ at $x = 4$. 3

(c) Find the nth derivative of $\sqrt{x}$. 4

(d) Evaluate any *two* of the following : 2½×2=5

(i) $\lim_{x \to 0} \frac{\tan x - x}{x - \sin x}$

(ii) $\lim_{x \to 0} x \log x$

(iii) $\lim_{x \to 0} x^{2 \sin x}$

UNIT—IV

7. (a) Integrate any *two* of the following : 4×2=8

(i) $\int \frac{x^2}{1-x^4} dx$

(ii) $\int \log(1+x)^{(1+x)} dx$

(iii) $\int \frac{dx}{(1-x)\sqrt{1+x}}$

(b) Express

$$\int_a^b x^2 \, dx$$

as the limit of a sum and evaluate it. 4

(c) Use the properties of definite integral to show that

$$\int_0^{\pi/2} \log \tan x \, dx = 0$$ 3

8. *(a)* Prove that

$$\int_{a-c}^{b-c} f(x+c)\,dx = \int_a^b f(x)\,dx$$ 2

(b) Evaluate :

$$\underset{n\to\infty}{\text{Lt}}\left[\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\cdots+\frac{n}{n^2+n^2}\right]$$ 4

(c) Obtain reduction formula for

$$\int_0^{\pi/2} \cos^n x\,dx$$

n being a positive integer greater than 1. Hence, evaluate

$$\int_0^1 x^2\sqrt{1-x^2}\,dx$$ 3+3

(d) Show that

$$\int_0^3 \frac{dx}{\sqrt{9-x^2}} = \frac{\pi}{2}$$ 3

UNIT—V

9. *(a)* State the degree and order of the following differential equations : 1+1

(i) $\dfrac{d^2y}{dx^2} = x^3\left(\dfrac{dy}{dx}\right)^2$

(ii) $y\left(\dfrac{dy}{dx}\right)^3 + 2x\dfrac{dy}{dx} - y = 0$

(b) Obtain the differential equation of the system of confocal conics

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$$

(λ is a variable parameter)

and hence, show that the system is self-orthogonal. 3+3

* * *

(c) Solve any *two* of the following : 3½×2=7

(i) $x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$

(ii) $x\,dy - y\,dx + a(x^2+y^2)\,dx = 0$

(iii) $\sec^2 y\dfrac{dy}{dx} + 2x\tan y = x^3$

10. *(a)* Find the general and singular solutions of $y = px + \sqrt{4p^2+1}$. 4

(b) Find $f(x)$, if $f'(x) = x\,f(x)$ and $f(0) = 1$. 3

(c) Find the equation of the curve whose slope at any point $(x,\ y)$ on it is xy and which passes through the point (0, 1). 3

(d) Solve any *one* of the following : 5

(i) $(D^2 - 9D + 20)y = x^2 e^{3x}$

(ii) $(D^2 - D - 2)y = \sin 2x$

(iii) $(D^2 - 2D + 1)y = 0;\ \ y = 0,\ \ Dy = 1$ when $x = 0$

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2016

(October)

MATHEMATICS

(Elective/Honours)

(Algebra—I and Calculus—I)

(GHS–11)

Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **five** questions, taking **one** from each Unit

UNIT—I

1. (a) If $A = \{a, b, c\}$, construct power set of A. 2

(b) Using Venn diagram, prove that for the sets X, Y and Z

$$(X - Y) - Z = X - (Y \cup Z)$$ 3

(c) Find the domain of the function

$$f(x) = \frac{1}{\sqrt{(3 - x)(x - 5)}}$$ 2

(d) Let

$$f(x)=\frac{|x|}{x} \text{ and } c\neq 0$$

be any real number. Show that

$$|f(c)-f(-c)|=2$$ 2

(e) Draw the graph of the function $y=[x]$, where $[x]$ denotes the greatest integer not exceeding x. 3

(f) A function f is defined as

$$f(x)=\begin{cases} x^2+2x+b & x\neq 0 \\ -3 & x=0 \end{cases}$$

For what value of b the function is continuous at $x=0$? 3

2. *(a)* Let A and B are two sets such that $n(A-B)=14+x$, $n(B-A)=3x$ and $n(A\cap B)=x$. If $n(A)=n(B)$, find the value of x. 2

(b) Using (ε, δ) definition, show that

$$\underset{x\to 2}{\mathrm{Lt}}\ \frac{2x^2-8}{x-2}=8$$ 4

(c) Let $f:\mathbb{R}\to\mathbb{R}$ is defined by $f(x)=3x+2$ and $g:\mathbb{R}\to\mathbb{R}$ is defined by

$$g(x)=\frac{x}{x^2+1}$$

Find $f\circ g$ and $g\circ g$. 2+2=4

(d) Let S be the set of all straight lines on a plane. A relation R is defined on S as lRm if and only if l is perpendicular to m $\forall\ l, m\in S$. Examine if R is *(i)* reflexive, *(ii)* symmetric and *(iii)* transitive. Is it an equivalence relation? Justify. 1½+1½+1½+½=5

UNIT—II

3. *(a)* Let I be the set of integers. A function $f:I\to I$ is defined by

$$f(x)=|x|\ \forall\, x\in I$$

Is f a one-one function? Justify your answer. 2

(b) If

$$A=\begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 5 \end{bmatrix} \text{ and } B=\begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

find $2A-3B$. 2

(c) Show that every square matrix A can be uniquely expressed as $P+iQ$, where P and Q are Hermitian matrices. 4

(d) Show that the system of equations

$$x+y+z=6$$
$$x+2y+3z=14$$
$$x+4y+7z=30$$

is consistent and solve them. 7

4. *(a)* A is a non-singular matrix of order 3. Show that $|adj\ A|=|A|^2$. 3

(b) Reduce the following matrix to normal form :

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$$

Hence find its rank. 5+1=6

(c) Obtain the inverse of the matrix

$$A=\begin{bmatrix} 1 & -2 & -1 \\ 2 & 3 & 1 \\ 0 & 5 & -2 \end{bmatrix}$$

by elementary operations. 6

UNIT—III

5. *(a)* Find the slope of the tangent to the curve $8y=x^3-12x+16$ at the point (0, 2). 3

(b) Find $\frac{dy}{dx}$, if

$$y=\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$$ 3

(c) If $y=\frac{1}{ax+b}$, find y_n. 3

(d) A circular plate of metal expands by heat so that its radius increases at the rate of a inches per second. At what rate is the surface area increasing when radius is b inches? 3

(e) Show that the equation $x^4-x^3-3=0$ has a real root between 1 and 2. Also state the theorem that you use. 1+2=3

6. *(a)* Find from definition the derivative of $\sin(\sqrt{x})$. 4

(b) Evaluate (by L'Hospital's rule) : 2+3=5

(i) $\lim_{x\to 0}\frac{e^x-1}{x}$

(ii) $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x^2}$

(c) Let $y = \tan^{-1} x$. Show that

(i) $(1+x^2)y_1 = 1$

(ii) $(1+x^2)y_{n+1} + 2nxy_n + n(n-1)y_{n-1} = 0$

2+4=6

UNIT—IV

7. (a) Evaluate (any *one*) : 3

(i) $\int \frac{dx}{1+\tan x}$

(ii) $\int \frac{dx}{(x-1)(x-2)}$

(b) Evaluate (any *one*) : 4

(i) $\int_0^1 x \log(1+2x)\, dx$

(ii) $\int_0^{2a} \sqrt{2ax - x^2}\, dx$

(c) Evaluate by the method of summation

$$\int_1^2 (x^2+2)\, dx$$

5

(d) Evaluate the following integral if convergent : 3

$$\int_0^\infty \frac{dx}{x^2+2x+2}$$

8. (a) Using the properties of definite integral, prove that

$$\int_0^\pi \frac{x \sin x}{1+\cos^2 x}\, dx$$

4

(b) Prove that

$$\lim_{n\to\infty}\left[\frac{1^2}{n^3+1^3} + \frac{2^2}{n^3+2^3} + \cdots + \frac{n^2}{2n^3}\right] = \frac{1}{3}\log 2$$

5

(c) If

$I_n = \int x^n \cos ax\, dx$ and $J_n = \int x^n \sin ax\, dx$

then prove that

$$aI_n = x^n \sin ax - nJ_{n-1}$$

4

(d) If

$$f(x) = \int_0^x \sqrt{t+t^6}\, dt \ (x > 0)$$

then find the value of $f'(2)$. 2

UNIT—V

9. (a) Obtain the differential equation of all circles passing through the origin and their centres lying on the X-axis. 3

(b) Solve (any *three*) : 3×3=9

(i) $(x+y)^2 dx = xy\, dy$

(ii) $(x^2 - yx^2)dy + (y^2 + xy^2)dx = 0$

(iii) $(x+y)dy+(y-x)dx=0$

(iv) $x\log x\frac{dy}{dx}+y=2\log x$

(v) $(1+x^2)\frac{dy}{dx}-xy=1$

(c) Show that the equation

$$(x^3-3x^2y+2xy^2)dx-(x^3-2x^2y+y^3)dy=0$$

is exact. 3

10. (a) Solve : 4

$$\frac{dy}{dx}+x\sin 2y=x^3\cos^2 y$$

(b) Solve : 4

$$p^2+p(x+y)+xy=0$$

(c) Obtain the complete primitive and singular solution of

$$(y+1)p-xp^2+2=0$$ 4

(d) Find the orthogonal trajectories of the family of curves $ay^2=x^3$. 3

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(GHS–11)

(Algebra—I and Calculus—I)

Marks : 75

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The figures in the margin indicate full marks for the questions

Answer **five** questions, taking **one** from each Unit

UNIT—I

1. *(a)* If $f(x+1) = x^2 - 3x + 2$, then show that $f(x) = x^2 - 5x + 6$. 4

(b) Let A, B, C are non-empty sets such that $A \times B = A \times C$. Show that $B = C$. 4

(c) If $f(x) = \dfrac{1}{1-x}$, then find the value of $f[f\{f(x)\}]$. 3

(d) Find the domain and the range of the function $f(x) = \frac{|x|}{x}$. Also draw the graph of $f(x)$. 2+2=4

2. *(a)* In an examination, 80 students secured first-class marks in Mathematics or English. Out of these, 50 students secured first-class marks in Mathematics only, and 10 students in English and Mathematics both. How many students secured first class in English only? 4

(b) Let $\mathbb{Z}$ be the set of all integers and a relation R is defined as

$$R = \{(a, b) \mid a - b \text{ is even}\}$$

Is it an equivalence relation? Justify. 4

(c) Show that the limit

$$\underset{x \to 2}{\text{lt}} \frac{|x-2|}{x-2}$$

does not exist. 4

(d) Evaluate : 3

$$\underset{x \to \infty}{\text{lt}} \frac{x^2 + 3x + 2}{x^3 + x - 4}$$

UNIT—II

3. *(a)* A mapping $f : \mathbb{N} \to \mathbb{N}$ is defined as $f(x) = x^3$. Show that f is one-one but not onto. 2

(b) Show that the matrix

$$A = \begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}$$

is nilpotent. Find its index. 3

(c) A and B are two non-singular matrices. Prove that $(AB)^{-1} = B^{-1}A^{-1}$. 4

(d) Find, applying elementary operations, the rank of the matrix

$$\begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$$

6

4. *(a)* Solve the system of equations, if consistent : 7

$$2x - y + 3z - 9 = 0$$
$$x + y + z - 6 = 0$$
$$x - y + z - 2 = 0$$

(b) Let A and B be any square matrices of same order. Show that $A + A^{\theta}$ is Hermitian and $A - A^{\theta}$ is skew-Hermitian. 4

(c) A and B are two symmetric matrices. Show that AB is symmetric if and only if A and B commute. 4

UNIT—III

5. *(a)* Let $f(x)$ be a continuous function in a closed interval and does not take the value 0 there. Prove that $f(x)$ keeps the same sign throughout the interval. 4

(b) Find the equation to the tangent to the curve $y = x^2 + 4x - 16$ which is parallel to the line $3x - y + 1 = 0$. 3

(c) Find the derivative of $x \log x$ from the first principle. 4

(d) If

$$y = \tan^{-1}\left(\frac{t}{\sqrt{1-t^2}}\right) \text{ and } x = \sec^{-1}\left(\frac{1}{2t^2 - 1}\right)$$

then show that $\frac{dy}{dx}$ is independent of t. 4

6. *(a)* A 20-feet ladder leans against a building while its base is drawn away from the wall at the rate 2 ft/sec. How fast is the top of the ladder descending when the ladder is inclined at an angle 60° to the horizontal? 3

(b) Evaluate any *two* of the following : 3×2=6

(i) $\lim_{x \to 0} \frac{e^x - e^{\sin x}}{x - \sin x}$

(ii) $\lim_{x \to 0} x \log(\tan x)$

(iii) $\lim_{x \to 0} (\sin x)^{\tan x}$

(c) Find y_n, if $y = \log x$. 3

(d) If $xy = \sin(x + y)$, then find $\frac{dy}{dx}$. 3

UNIT—IV

7. *(a)* Evaluate any *two* of the following : 2½×2=5

(i) $\int \sec^3 x \, dx$

(ii) $\int \frac{e^x}{x}(1 + x \log x) \, dx$

(iii) $\int \frac{x^2}{x^2 - 4} \, dx$

(b) Obtain a reduction formula for $\int \sin^m x \cos^n x \, dx$. Using this formula, obtain the value of $\int_0^{\pi/2} \sin^6 x \cos^8 x \, dx$. 4+2=6

(c) Show that

$$\int_0^{\infty} \frac{dx}{(x^2 + a^2)(x^2 + b^2)} = \frac{\pi}{2ab(a+b)} \quad a, b > 0$$

4

8. (a) Using the properties of definite integral, show that

$$\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx = \frac{\pi}{8}\log 2$$ 5

(b) Find the value of $\int_0^1 x^3 dx$ by the method of summation. 4

(c) Evaluate : 4

$$\lim_{n\to\infty}\left[\frac{1+2^{10}+3^{10}+\cdots+n^{10}}{n^{11}}\right]$$

(d) Show that

$$\int_0^2 |1-x|\,dx = 1$$ 2

UNIT—V

9. (a) Show that $y = e^{-x}(A\cos x + B\sin x)$ is the solution of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$$ 3

(b) Prove that for any straight line $\frac{d^2y}{dx^2} = 0$. 2

(c) Solve : 4

$$\frac{dy}{dx} + \frac{y}{x}\log y = \frac{y}{x^2}(\log y)^2$$

(d) Solve any *two* of the following : 3×2=6

(i) $(6x-8y-5)\,dy = (3x-4y-2)\,dx$

(ii) $(e^x+1)y\,dy + (y+1)e^x dx = 0$

(iii) $x^2 dy = (x^2+5xy+4y^2)\,dx$

(iv) $x\,dx + y\,dy + (x^2+y^2)\,dy = 0$

10. (a) Solve any *two* of the following : 4×2=8

(p stands for $\frac{dy}{dx}$)

(i) $x - yp = ap^2$

(ii) $p^2 - p(e^x + e^{-x}) + 1 = 0$

(iii) $y = (1+p)x + ap^2$

(b) Find the general and singular solution of $y = px + \sqrt{a^2p^2+b^2}$. 4

(c) Show that the equation to the curve whose slope at any point is equal to $y+2x$ and which passes through origin is $y = 2(e^x - x - 1)$. 3

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(October)

MATHEMATICS

(Elective/Honours)

(GHS–11)

(Algebra—I and Calculus—I)

Marks : 75

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Answer **five** questions, taking **one** from each Unit

UNIT—I

1. *(a)* Find the domain of the function

$$f(x)=\frac{1}{\sqrt{|x|-x}}$$ 3

(b) If

$$f(x)=\frac{1+e^x}{1-e^x}, \quad x\neq 0$$

show that $f(x)$ is an odd function. 3½

(c) A function f is defined as follows :

$$f(x) = 2x + 3 \quad \text{for} \quad x > 2$$
$$= 3x + 4 \quad \text{for} \quad x \le 2$$

Examine if $f(x)$ is continuous at $x = 2$. Draw the graph of $f(x)$. 3½+1=4½

(d) A and B are two sets as given below :

$$A = \{1, 2, 3\}, \ B = \{x, y\}$$

Obtain $A \times B$ and $B \times A$. 2+2=4

2. *(a)* Using the definition of limit at ∞, show that

$$\lim_{x \to \infty} \frac{x}{1+x} = 1$$ 4

(b) Let $f, g : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = e^x$ and $g(x) = \sin x$. Obtain $f \circ g$ and $g \circ f$. Is $f \circ g = g \circ f$? Discuss the continuities of $f \circ g$ and $g \circ f$. 3+½+1½=5

(c) A relation R is defined on $\mathbb{R}$, the set of real numbers, as follows :

aRb when $a \ne b$. Examine if the relation is *(i)* reflexive, *(ii)* symmetric and *(iii)* transitive. 3

(d) Prove that for any two sets A and B, $(A \cap B)' = A' \cup B'$, where A' = complement of A. 3

UNIT—II

3. *(a)* Give example of a mapping $f : A \to B$ such that f is—

(i) one-one but not onto;

(ii) onto but not one-one;

(iii) one-one and onto;

(iv) neither one-one nor onto. 4

(b) If A and B are two matrices such that $AB = A$ and $BA = B$, show that A' and B' are idempotent. ($A' \equiv$ transpose of A). 4

(c) Examine if the following system of equations is consistent and if so, find the solution : 7

$$x + y + z = 6$$
$$x - y + z = 2$$
$$2x + y - z = 1$$

4. *(a)* If A is a non-singular square matrix of order n, prove that $|adj\,A| = |A|^{n-1}$. 3

(b) Reduce the following matrix into normal form and hence obtain its rank : 8

$$\begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}$$

(c) Prove that every square matrix is uniquely expressible as the sum of a Hermitian and a skew-Hermitian matrices. 4

UNIT—III

5. (a) Using definition, find the derivative of x^2+7x+9. 4

(b) Find $\frac{dy}{dx}$ (any *one*), when 3

(i) $y=\cos^{-1}\frac{1-x^2}{1+x^2}$;

(ii) $y=\frac{1-\sin x}{1+\sin x}$.

(c) If $y=x^{\tan x}+(\sin x)^{\cos x}$, find $\frac{dy}{dx}$. 5

(d) If the rate of change of y with respect to x is 5 and x is changing at 3 units per second, how fast is y changing? 3

6. (a) If $y=\sin^{-1}x$, prove that

$$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-n^2y_n=0$$ 5

(b) Evaluate any *one* of the following : 3

(i) $\lim_{x\to\frac{\pi}{4}}\frac{1-\tan x}{1-\cot x}$

(ii) $\lim_{x\to 1}\left[\frac{x}{x-1}-\frac{1}{\log x}\right]$

(c) When is a function said to be uniformly continuous in an interval? Show that the function $f(x)=x^2$ is uniformly continuous in $[-1, 1]$. 1+3=4

(d) Find the derivative of $x^{\sin^{-1}x}$ with respect to $\sin^{-1}x$. 3

UNIT—IV

7. (a) Evaluate any *one* of the following : 3½

(i) $\int\frac{1}{x(x+1)^2}dx$

(ii) $\int\frac{dx}{5+4\cos x}$

(b) Show that (any *one*) 3½

(i) $\int_e^{e^2}\frac{dx}{x\log x}=\log 2$;

(ii) $\int_0^{1/2}\sin^{-1}x\,dx=\frac{\pi}{12}-1+\frac{\sqrt{3}}{2}$.

(c) Show that

$$\int\tan^5 x\,dx=\frac{\tan^4 x}{4}-\frac{\tan^2 x}{2}+\log|\sec x|$$ 4

(d) Prove that

$$\lim_{n\to\infty}\left[\frac{1}{n}+\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+\cdots+\frac{1}{8n}\right]=\frac{3}{8}$$ 4

8. *(a)* If

$$f(x) = \cos x \quad \text{for} \quad -\tfrac{\pi}{2} \le x \le 0$$
$$= \sin x \quad \text{for} \quad 0 < x \le \tfrac{\pi}{2}$$

show that

$$\int_{-\pi/2}^{\pi/2} f(x)\, dx = 2$$ 3

(b) Using the properties of definite integral, show that

$$\int_0^{\pi} \frac{x\,dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{\pi^2}{2ab}$$ 4

(c) Examine the convergence of

$$\int_0^1 \frac{dx}{x^{2/3}}$$ 4

(d) Using the definition of definite integral, evaluate

$$\int_a^b e^x dx$$ 4

UNIT—V

9. *(a)* Show that the differential equation whose general solution is $y = ax + bx^2$ is

$$y = x\frac{dy}{dx} - \frac{1}{2}x^2 \frac{d^2y}{dx^2}$$ 3

(b) Solve any *two* of the following : 2½×2=5

(i) $xy^2 dy - y^3 dx + y^2 dy = dx$

(ii) $x^2 dy + (xy + 2y^2)\, dx = 0$

(iii) $x\frac{dy}{dx} = y + \cos^{-1}\frac{1}{x}$

(c) Solve any *one* of the following : 3

(i) $x \log x \frac{dy}{dx} + y = 2 \log x$

(ii) $x\frac{dy}{dx} = y + e^{1/x}$

(d) Solve any *one* of the following : 4

(i) $(1 + y^2)\, dx - (\tan^{-1} y - x)\, dy = 0$

(ii) $\frac{dy}{dx} + \frac{y}{x}\log y = \frac{y}{x^2}(\log y)^2$

10. *(a)* Solve any *two* of the following equations : 3½×2=7

(i) $p^2 - p(x + y) + xy = 0$

(ii) $y = (1 + p)x + p^2$

(iii) $y = yp^2 + 2px$

(b) Obtain the complete primitive and singular solution of

$$(y+1)p - xp^2 + 2 = 0$$ 4

(c) Find the orthogonal trajectories of the curve

$$x^2 + y^2 + 2gx + c = 0$$

where g is a parameter. 4

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1/EH–29 (i) (Syllabus–2015)

2019

(October)

MATHEMATICS

(Elective/Honours)

(GHS–11)

(Algebra–I & Calculus–I)

Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **five** questions, taking **one** from each Unit

UNIT—I

1. *(a)* Prove that for any two sets A and B $(A \cup B)^c = A^c \cap B^c$, where A^c is the complement of A. 3

(b) If $f(x+3) = 2x^2 - 3x + 1$, then find $f(x+1)$. 4

(c) Find the domain and range of the function $f(x) = \frac{x^2}{x}$. Also draw the graph of $f(x)$. 2+3=5

(d) Give an example of a relation which is—

(i) reflexive, symmetric but not transitive;

(ii) symmetric and transitive but not reflexive;

(iii) reflexive and anti-symmetric. 1×3=3

2. *(a)* Using ε-δ definition of limit, show that

$$\lim_{x\to 3}\frac{x^2-9}{x-3}=6$$ 4

(b) For what value of *a*, $f(x)=2ax+3$, $x\neq 2$ and $f(2)=3$ is continuous at $x=2$? 3

(c) Let $\mathbb{Z}$ be the set of all integers and a relation *R* is defined as

$$R=\{(a,b):a-b \text{ is even}\}$$

Is it an equivalence relation? Justify. 5

(d) In a group of 1000 people who can speak Khasi or Bengali; there are 750 who can speak Khasi and 400 who can speak Bengali. How many can speak Khasi only? How many can speak Bengali only? How many can speak both Khasi and Bengali? 3

UNIT—II

3. *(a)* Let $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x)=x^2$, $x\in\mathbb{R}$. Examine if *f* is—

(i) injective;

(ii) surjective. 1½+1½=3

(b) Give examples of—

(i) matrices *A* and *B* such that $AB\neq BA$;

(ii) matrices *A* and *B* such that $AB=0$ but $A\neq 0$, $B\neq 0$. 2+2=4

(c) Solve the following system of linear equations with the help of Cramer's rule : 4

$$x+2y+3z=6$$
$$2x+4y+z=7$$
$$2x+2y+9z=14$$

(d) Show that every square matrix is uniquely expressible as the sum of a symmetric matrix and a skew-symmetric matrix. 4

4. *(a)* If

$$A=\begin{bmatrix}3 & 1 & -1\\ 0 & 1 & 2\end{bmatrix}$$

then find AA' and $A'A$, where A' is the transpose of matrix *A*. 2+2=4

(b) Find the rank of the matrix

$$A=\begin{bmatrix}2 & -2 & 0 & 6\\ 4 & 2 & 0 & 2\\ 1 & -1 & 0 & 3\\ 1 & -2 & 1 & 2\end{bmatrix}$$

by reducing it to normal form. 6

(c) Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ -2 & 1 & 1 \end{bmatrix}$$

by elementary row operations. 5

UNIT—III

5. (a) Find the derivative of $\sin x$, $x > 0$ from the first principle. 4

(b) Find $\frac{dy}{dx}$ of the following (any *one*) : 4

(i) $x^y y^x = 1$

(ii) $y = \tan^{-1}\left(\frac{a+bx}{b-ax}\right)$

(c) If $\tan y = \frac{2t}{1-t^2}$ and $\sin x = \frac{2t}{1+t^2}$, then find $\frac{dy}{dx}$. 4

(d) Find the slope of the tangent line at the point $(0, 2)$ of the curve $8y = x^3 - 12x + 16$. 3

6. (a) State Leibnitz's theorem on the *n*th derivative of the product of two functions. Find y_n, if $y = \sqrt{x}$. 2+2=4

(b) If $y = (x^2 - 1)^n$, then prove that

$$(x^2 - 1)y_2 - (n-1)2xy_1 - 2ny = 0$$

and hence

$$(x^2 - 1)y_{n+2} + 2xy_{n+1} - n(n+1)y_n = 0$$ 5

(c) Use L' Hospital's rule to evaluate the following limits (any *one*) : 3

(i) $\underset{x\to\infty}{\text{Lt}} \frac{x^3}{e^x}$

(ii) $\underset{x\to 0}{\text{Lt}} \frac{\log(x^2)}{\log(\cot^2 x)}$

(d) Water is running into a conical reservoir, 10 cm deep and 5 cm radius at the rate 1.5 cc per minute. At what rate is the water level rising when the water is 4 cm deep? 3

UNIT—IV

7. (a) Evaluate any *two* of the following : 2×2=4

(i) $\int \frac{x}{\sqrt{x+1}}\,dx$

(ii) $\int e^x \left(\frac{1-x}{1+x^2}\right)^2 dx$

(iii) $\int \frac{dx}{(\sin x + \cos x)^2}$

(b) Show that any *one* of the following : 3

(i) $\int_0^{\frac{\pi}{2}} \log \sin x \, dx = \frac{\pi}{2} \log \frac{1}{2}$

(ii) $\int_0^{\infty} \frac{dx}{(x+1)(x+2)} = \log 2$

(c) Let n be a positive integer and let $I_n = \int x^n e^{ax} dx$. Derive the reduction formula

$$I_n = \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1}$$

Hence find $\int x^5 e^{ax} dx$. 2+2=4

(d) Evaluate the following : 4

$$\underset{n\to\infty}{\text{Lt}} \left[\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \ldots + \frac{n}{n^2+n^2} \right]$$

8. *(a)* Express $\int_0^1 (ax+b)\,dx$ as the limit of a sum and evaluate it. 4

(b) Using the properties of definite integral, show that

$$\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx = \frac{\pi^2}{4}$$ 4

(c) Find $\int_0^3 f(x)\,dx$, where $f(x) = 2x$, when $0 \le x \le 2$ and $f(x) = x^2$, when $2 \le x \le 3$. 3

(d) Evalaute $\int_{-\infty}^{\infty} xe^{-x^2} dx$ if it converges. 4

UNIT—V

9. *(a)* Show that $Y = Ae^{2x} + Be^{-2x}$ is the solution of the differential equation

$$\frac{d^2y}{dx^2} - 4y = 0$$ 3

(b) Solve any *two* of the following : 2½×2=5

(i) $\frac{dy}{dx} = \frac{x^2+x+1}{y^2+y+1}$

(ii) $x + y\frac{dy}{dx} = 2y$

(iii) $(2x - y + 1)\,dx + (2y - x - 1)\,dy = 0$

(c) Solve : $\frac{dy}{dx} + \frac{y}{x} = y^2$ 3

(d) Find the equation of orthogonal trajectories of the family of curves $\frac{x^2}{a^2} + \frac{y^2}{a^2+\lambda^2} = 1$, where λ is a parameter. 4

10. *(a)* Solve any *two* of the following : 4×2=8

(i) $p^2 + p - 6 = 0$

(ii) $p^2 - (a+b)p + ab = 0$

(iii) $p^2 - p(e^x + e^{-x}) + 1 = 0$

$\left[p \text{ stands for } \dfrac{dy}{dx}\right]$

(b) Find the general and singular solution of $y = px + ap(1-p)$. 4

(c) Find the equation of the curve whose slope at any point (x, y) is xy and which passes through the point $(0, 1)$. 3

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